

UNIT-II

Sets and Relations.

Relations

Introduction:-

The elements of a set may be related to one another, for example, in the set of natural numbers there is the 'less than' relation between the elements. The elements of one set may also be related to the elements of another set.

Binary Relation

A binary relation between two sets A and B is a rule R which decides, for any elements, whether a is in relation R to b . If so, we write $a R b$. If a is not in relation R to b , then we shall write $a \not R b$.

We can also consider $a R b$ as the ordered pair (a, b) in which case we can define a binary relation from A to B as a subset of $A \times B$. This subset is denoted by the relation R .

In general any set of ordered pairs defines a binary relation.

For example, the relation of father to his child is $F = \{(a, b) / a \text{ is the father of } b\}$ in this relation F , the first member is the name of the father and the second is the name of the child. The definition of relation permits any set of ordered pairs to define a relation.

for example: the set S given by

$$S = \{(1, 2), (3, a), (b, a), (b, \text{Joe})\}$$

Definition

The domain D of a binary relation S is the set of all first elements of the ordered pairs in the relation (i.e.,) $D(S) = \{a / \exists b \text{ for which } (a, b) \in S\}$

The range R of a binary relation S is the set of all second elements of the ordered pairs in the relation (i.e.) $R(S) = \{b / \exists a \text{ for which } (a, b) \in S\}$

for example

for the relations $S = \{(1, 2), (3, a), (b, a), (b, \text{Joe})\}$

$$D(S) = \{1, 3, b, b\} \text{ and}$$

$$R(S) = \{2, a, a, \text{Joe}\}$$

Let x and y be any two sets. A subset of the cartesian product $x \times y$ defines a relation, say c . For any such relation c , we have $D(c) \subseteq X$ and $R(c) \subseteq Y$; and the relation c is said to from x to y . If $y = X$, then c is said to be a relation form X to X . In such case, c is called a relation in X . Thus any relation in X is a subset of $X \times X$. The set $X \times X$ is called a universal relation in X , while the empty set which is also a subset of $X \times X$ is called a void relation in X .

for example

Let L denote the relation "less than or equal to" and D denote the relation.

"divides" where $x \mid y$ means " x divides y ".

Both L and D are defined on the set $\{1, 2, 3, 4\}$.

$$L = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

$$D = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

$$LCD = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

$$= D$$

Properties of Binary Relations:

Definition: A binary relation R in a set X is reflexive if, for every $x \in X$, $x R x$. That is $(x,x) \in R$, or R is reflexive in X $O(x)(x \in X \rightarrow x R x)$.

for example

- The relation F is reflexive in the set of real numbers.
- The set inclusion is reflexive in the family of all subsets of a universal set.
- The relation equality of set is also reflexive.
- The relation is parallel in the set lines in a plane.
- The relation of similarity in the set of triangles in a plane is reflexive.

Definition: A relation \mathcal{R} in a set X is symmetric if for every x and y in X , whenever $x \mathcal{R} y$, then $y \mathcal{R} x$ (i.e) \mathcal{R} is symmetric in X $\delta(x)(y)(x \in X \wedge y \in X \wedge x \mathcal{R} y \Rightarrow y \mathcal{R} x)$

for example

- The relation equality of set is symmetric
- The relation of similarity in the set of triangles in a plane is symmetric.
- the relation of being a sister is not symmetric in the set of all people.
- However, in the set females it is symmetric

Definition:- A relation \mathcal{R} in a set X is transitive if for every x, y , and z are in X whenever $x \mathcal{R} y$ and $y \mathcal{R} z$, then $x \mathcal{R} z$ (i.e) \mathcal{R} is transitive in X $\delta(x)(y)(z)(x \in X \wedge y \in X \wedge z \in X \wedge x \mathcal{R} y \wedge y \mathcal{R} z \Rightarrow x \mathcal{R} z)$

for example:-

- the relations $<$, \in , $>$, ' \neq ' and $=$ are transitive in the set of real numbers
- The relation \cup , \cap , \subseteq , \supseteq and equality are also transitive in the family of sets.
- The relation of similarity in the set of triangles in a plane is transitive,

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Definition:- A relation \mathcal{R} in a set X is irreflexive if, for every $x \in X$, $(x, x) \notin \mathcal{R}$

for example

- The relation $<$ is irreflexive in the set of all real numbers.
- The relation proper inclusion \subsetneq is irreflexive in the set of all nonempty subsets of a universal set.
- Let $X = \{1, 2, 3\}$ and $S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is neither irreflexive nor reflexive.

Definition:-

A relation \mathcal{R} in a set X is anti symmetric if, for every x and y in X , whenever $x R y$ and $y R x$, then $x = y$.

Symbolically, $(\forall x)(\forall y)(x \in X \wedge y \in X \wedge x R y \wedge y R x \rightarrow x = y)$

for example.

- The relations \neq and $=$ are anti symmetric.
- the relation I is anti symmetric in set of subsets.
- The relation "divides" is anti symmetric in set of real numbers.
- consider the relation "is a son of" on the male children in a family . Evidently the relation is not symmetric, transitive and reflexive.
- The relation "is a divisor of" is reflexive and transitive but not symmetric on the set of natural numbers.

- consider the set H of all human beings. let R be a relation "is married to" R is symmetric.
- let I be the set of integers. R on I is defined as $a R b$ if $a - b$ is an even number. R is an reflexive, symmetric and transitive, equivalence Relation;

Definition: A relation R in a set A is called an equivalence relation if

- $a Ra$ for every i.e. R is reflexive
- $a R b \Rightarrow b R a$ for every $a, b \in A$ i.e. R is symmetric
- $a R b$ and $b R c \Rightarrow a R c$ for every $a, b, c \in A$, i.e. R is transitive.

for example

- The relation equality of numbers on set of real numbers.
- The relation being parallel on a set of lines in a plane.

Problem 1: let us consider the set T of triangles in a plane. Let us define a relation R in T as $R = \{(a, b) / (a, b \in T \text{ and } a \text{ is similar to } b)\}$ we have to show that relation R is an equivalence relation.

Solution

- A triangle a similar to itself. $a Ra$.

- If the triangle a is similar to the triangle b , then triangle b 's similar to the triangle a then
 $a R b \Rightarrow b Ra$
- If a is similar to b and b is similar to c , then a is similar to c (i.e) $a R b$ and $b R c \Rightarrow a R c$.

Problem 2: Let $X = \{1, 2, 3, \dots, 7\}$ and $R = \{(x, y) | x - y \text{ is divisible by } 3\}$ show that relation is an equivalence relation.

Solution: for any $a \in X$, $a - a$ is divisible by 3,

Hence aRa , R is reflexive.

for any $a, b \in X$, if $a - b$ is divisible by 3,
 then $b - a$ is also divisible by 3.

R is symmetric.

for any $a, b, c \in X$, if $a R b$ and $b R c$, then $a - b$ is divisible by 3 and $b - c$ is divisible by 3, so that $(a - b) + (b - c)$ is also divisible by 3. hence $a - c$ is also divisible by 3. Thus R is transitive.

\therefore Hence R is equivalence.

Problem 3 Let Z be the set all integers. let m be a fixed integer. Two integers a and b are said to be congruent modulo m if and only if m divided $a - b$, in which case we write $a \equiv b \pmod{m}$. This relation is called the relation of congruence modulo m and we can show that is an equivalence relation.

Solution :

- $a - a = 0$ and m divides $a - a$ (i.e.) aRa ,
 $(a, a) \in R$ is reflexive.

- $a R b \Rightarrow m$ divides $a - b$
 m divides $b - a$
 $b \equiv a \pmod{m}$

$b R a$
 that is R is symmetric,

- $a R b$ and $b R c \Rightarrow a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$
- $a m$ divides $a - b$ and m divides $b - c$
- $(a - b) + (b - c) = km + lm$
- $a - c = (k+1)m$
- $a \equiv c \pmod{m}$
- $a R c$

$\therefore R$ is transitive.

Hence the congruence relation is an equivalence relation.

Equivalence classes:

Let R be an equivalence relation on a set A ,
 for any $a \in A$, the equivalence class generated
 by a is the set of all elements $b \in A$ such
 $a R b$ and is denoted $[a]$. It is also called the
 R -equivalence class and denoted by $a \sim_A$, i.e.,
 $[a] = \{b \in A \mid b Ra\}$

Let \mathbb{Z} be the set of integer and \mathcal{R} be the relation called "congruence modulo 3"

defined by $= \{(x,y) / x \equiv y \pmod{3} \text{ or } (x-y) \text{ is divisible by 3}\}$

Then the equivalence classes are

$$[0] = \{ \dots -6, -3, 0, 3, 6, \dots \}$$

$$[1] = \{ \dots -5, -2, 1, 4, 7, \dots \}$$

$$[2] = \{ \dots -4, -1, 2, 5, 8, \dots \}$$

Composition of binary relations:

Definition:

Let \mathcal{R} be a relation from x to y and s be a relation from y to z . Then the relation $R \circ S$ is given by $R \circ S = \{(x,z) / \exists x \in X \exists z \in Z \exists y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}$ is called the composite relation of R and S .

The operation of obtaining $R \circ S$ is called the composition of relations.

Example: Let $R = \{(1,2), (3,4), (2,2)\}$ and

$$S = \{(4,1), (2,5), (3,1), (1,3)\}$$

$$\text{then } R \circ S = \{(1,5), (3,2), (2,5)\} \text{ and}$$

$$S \circ R = \{(4,2), (3,2), (1,4)\}$$

it is to be noted that $R \circ S \neq S \circ R$.

$$\text{Also } R^0(S^0)^T = (R^0S^0)^0T = R^0S^0T$$

Note: we write $R \circ R$ as R_2 , $R \circ R \circ R$ as R_3 and so on.

Definition

Let R be a relation from X to Y , a relation R' from Y to X is called the converse of R , where the ordered pairs of R' are obtained by interchanging the numbers in each of the ordered pairs of R . This means for $x \in X$ and $y \in Y$, that $xRy \Leftrightarrow yRx$.

Then the relation R' is given by $R' = \{(x, y) / (y, x) \in R\}$ is called the converse of R .

Example:

$$\text{Let } R = \{(1, 2), (3, 4), (2, 2)\}$$

$$\text{then } R' = \{(2, 1), (4, 3), (2, 2)\}$$

Note: if R is an equivalence relation, then R' is also an equivalence relation.

Definition Let X be any finite set and R be a relation in X . The relation

$R^+ = R \cup R^2 \cup R^3 \cup \dots$ in X , is called the transitive closure of R in X .

Example: Let $R = \{(a, b), (b, c), (c, a)\}$

$$\text{Now } R_2 = R \circ R = \{(a, b), (b, c), (c, a)\}$$

$$R_3 = R_2 \circ R = \{(a, a), (b, b), (c, c)\}$$

$$R_4 = R_3 \circ R = \{(a, b), (b, c), (c, a)\} = R$$

$$R_5 = R_3 \circ R_2 = R_2 \text{ and so on.}$$

$$\text{Thus, } R^+ = R \cup R_2 \cup R_3 \cup R_4 \cup \dots$$

$$= R \cup R_2 \cup R_3.$$

$$= \{(a, b), (b, c), (c, a), (a, c), (b, a), (c, b), (b, b), (c, c)\}$$

We see that R^+ is a transitive relation containing R . In fact, it is the smallest transitive relation containing R .

partial ordering Relations:-

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Definitions:-

A binary relation \mathcal{R} in a set P is called partial order relation or partial ordering in P iff \mathcal{R} is reflexive, anti symmetric, and transitive.

A partial order relation is denoted by the symbol \in , if \in is a partial ordering on P . then the ordered pair (P, \in) is called a partially ordered set or a poset.

- Let R be the set of real numbers. The relation "less than or equal to" or \leq , is a partial ordering on R .

- Let X be a set and $\mathcal{P}(X)$ be its power set. The relation subset, \subseteq on X is partial ordering.

- Let S_n be the set of divisors of n . The relation D means "divides" on S_n , is partial ordering on S_n .

In a partially ordered set (P, \in) , an element $y \in P$ is said to cover an element $x \in P$ if $x < y$ and if there does not exist any element $z \in P$ such that $x < z$ and $z < y$; that is, y covers x $\Leftrightarrow (x < y \wedge \forall z \in P \quad x < z \Rightarrow z = y)$

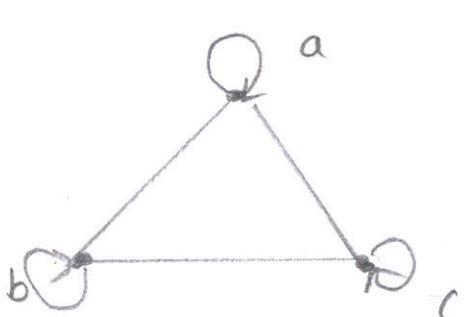
A partial order relation \in on a set P can be represented by means of a diagram known as a Hasse diagram or partial order set diagram of (P, \in) . In such a diagram, each element is represented by a small circle or a dot. The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$, and a line is drawn between x and y if y covers x .

If $x \prec y$ but y does not cover x , then x and y are not connected directly by a single line however, they are connected through one or more elements of P .

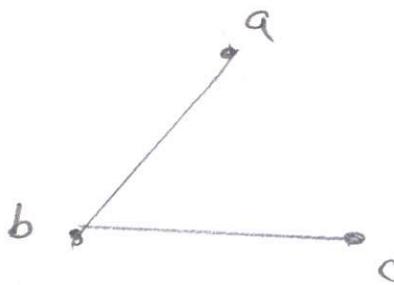
Hasse Diagram:

A Hasse diagram is a digraph for a poset which does not have loops and arcs implied by the transitivity.

Example 10: for the relation $\{ \langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, c \rangle \}$ on set $\{a, b, c\}$, the hasse diagram has the arcs $\{ \langle a, b \rangle, \langle b, c \rangle \}$ as shown below.

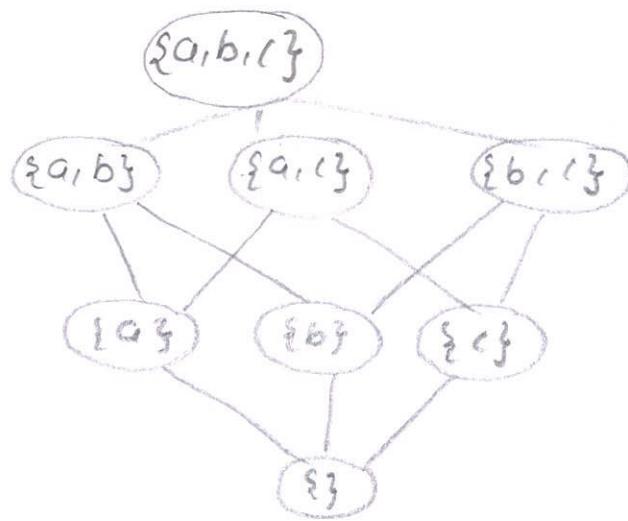


Digraph for partial order



Hasse Diagram

Ex: Let A be a given finite set and $r(A)$ its power set. Let I be the subset relation on the elements of $r(A)$. Draw Hasse diagram of $(r(A), I)$ for $A = \{a, b, c\}$



functions:

Introduction

A function is a special type of relation. It may be considered as a relation in which each element of the domain belongs to only one ordered pair in the relation. Thus a function from A to B is a subset of $A \times B$ having the property that for each $a \in A$, there is one and only one $b \in B$ such that $(a, b) \in f$.

Definition

Let A and B be any two sets. A relation f from A to B is called a function if for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$.

Note that the definition of function requires that a relation must satisfy two additional conditions in order to qualify as a function

The first condition is that every $a \in A$ must be related to some $b \in B$, i.e. the domain of f must be A and not merely subset of A.

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The second requirement of uniqueness can be expressed as $(a, b) \in f \wedge (b, c) \in f \Rightarrow b = c$

Intuitively, a function from a set A to a set B is a rule which assigns to every element of A, a unique element of B. If $a \in A$, then the unique element of B assigned to a under f is denoted by $f(a)$. The usual notation for a function f from A to B is $f: A \rightarrow B$ denoted by $a \in A \rightarrow f(a)$ where $a \in A, f(a)$ is called the image of a under f and a is called pre-image of $f(a)$.

- Let $X = Y = \mathbb{R}$ and $f(x) = x^2 + 2$. $Df = \mathbb{R}$ and $Rf \subset \mathbb{R}$
- Let X be the set of all statements in logic and let $Y = \{\text{true}, \text{false}\}$.

A mapping $F: X \rightarrow Y$ is a function

- A program written in level language is mapped into a machine language by a compiler. Similarly, the output from a compiler is a function of its input.

- Let $X = Y = \mathbb{R}$ and $f(x) = x^2$ is a function from $X \rightarrow Y$, and $g(x_2) = x$ is not a function from $X \rightarrow Y$

A mapping $f: A \rightarrow B$ is called one-to-one (injective or 1-1) if distinct elements of A are mapped into distinct elements of B. (i.e.) f is one-to-one if

④ mapping $f: A \rightarrow B$ is called one-to-one

$a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$ or equivalently $f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2$

for example $f: N \rightarrow N$ given by $f(x) = x$ is 1-1 where N is the set of natural numbers

④ mapping $f: A \rightarrow B$ is called into if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$. i.e. if every element of B has a pre-image in A , otherwise it is called into.

for example $f: R \rightarrow R$ given by $f(x) = x + 1$ is bijective.

Definition: ④ mapping $f: R \rightarrow b$ is called a constant mapping if, for all $a \in A$, $f(a) = b$, a fixed element.

for example $f: Z \rightarrow Z$ given by $f(x) = 0$, for all $x \in Z$ is a constant mapping.

Definition

④ mapping $f: A \rightarrow A$ is called the identity mapping of A if $f(a) = a$, for all $a \in A$, usually it is denoted by I_A or simply I .

Definition

④ mapping $f: A \rightarrow$

composition of functions;

If $f: A \rightarrow B$ and $g: B \rightarrow C$ are two functions then the composition of functions f and g , denoted by gof , is the function given by $gof: A \rightarrow C$ and is given by

$$gof = \{(a, c) / a \in A \text{ and } c \in C \text{ such that } b \in B; f(a) = b \text{ and } g(b) = c\}$$

Example 1:

Consider the sets $A = \{1, 2, 3\}$, $B = \{a, b\}$ and $C = \{x, y\}$.

Let $f: A \rightarrow B$ be defined by $f(1) = a$; $f(2) = b$ and $f(3) = b$ and

let $g: B \rightarrow C$ be defined by $g(a) = x$ and $g(b) = y$

(i.e.) $f = \{(1, a), (2, b), (3, b)\}$ and $g = \{(a, x), (b, y)\}$.

Then $gof: A \rightarrow C$ is defined by

$$(gof)(1) = g(f(1)) = g(a) = x$$

$$(gof)(2) = g(f(2)) = g(b) = y$$

$$(gof)(3) = g(f(3)) = g(b) = y$$

$$\text{i.e. } gof = \{(1, x), (2, y), (3, y)\}$$

If $f: A \rightarrow A$ and $g: A \rightarrow A$, where $A = \{1, 2, 3\}$, are given by

$$f = \{(1, 2), (2, 3), (3, 1)\} \text{ and } g = \{(1, 3), (2, 2), (3, 1)\}$$

then $gof = \{(1,2), (2,1), (2,3)\}$, $fog = \{(1,1), (2,3), (3,2)\}$

$f \circ f = \{(1,3), (2,1), (3,2)\}$ and $g \circ g = \{(1,1), (2,2), (3,3)\}$

Example 2: let $f(x) = x+z$, $g(x) = x-z$ and $h(x) = 3x$

for $x \in R$, where R is the set of real numbers.

then $f \circ f = \{(x, x+4) / x \in R\}$

$f \circ g = \{(x, x) / x \in X\}$

$g \circ f = \{(x, x) / x \in X\}$

$g \circ g = \{(x, x-4) / x \in X\}$

$h \circ g = \{(x, 3x-6) / x \in X\}$

$h \circ f = \{(x, 3x+6) / x \in X\}$

Inverse functions:-

Let $f: A \rightarrow B$ be a one-to-one and onto mapping. Then, its inverse, denoted by f^{-1} is given by $f^{-1} = \{(b,a) / (a,b) \in F\}$ clearly $f^{-1}: B \rightarrow A$ is one-to-one and onto.

Also we observe that $f \circ f^{-1} = IB$ and $f^{-1} \circ f = IA$. If f^{-1} exists then f is called invertible.

for example; let $f: R \rightarrow R$ be defined by $f(x) = x+2$
then $f^{-1}: R \rightarrow R$ is defined by $f^{-1}(x) = x-2$.

Theorem: Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two one to one and onto functions, then gof is also one to one and onto function.

Proof

Let $f: x \rightarrow y$ & $y \rightarrow z$ be two one to one to onto functions let $x_1, x_2 \in X$.

- $g \circ f(x_1) = g \circ f(x_2)$,
- $g(f(x_1)) = g(f(x_2))$,
- $g(x_1) = g(x_2)$ since [f is 1-1]

$x_1 = x_2$ since [g is 1-1]

so that $g \circ f$ is 1-1.

by the definition of composition $g \circ f: x \rightarrow z$ is a function we have to prove that every element of $Z \setminus z$ is an image element for some $x \in X$ under $g \circ f$. since g is onto $\exists y \in Y : g(y) = z$ and f is onto from X to Y .

$\exists x \in X : f(x) = y$,

$$\text{Now, } g \circ f(x) = g(f(x))$$

$$= g(y) \quad [\text{since } f(x) = y]$$

$$= z \quad [\text{since } g(y) = z]$$

which shows that $g \circ f$ is onto

Theorem $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

(i.e) the inverse of a composite function can be expressed in terms of the composition of the inverses in the reverse order

Proof,

$f: A \rightarrow B$ is one to one and onto.

$g: B \rightarrow C$ is one to one and onto

$gof: A \rightarrow C$ is also one to one and onto

$p(gof)^{-1}: C \rightarrow A$ is one to one and onto.

Let $a \in A$, then there exists an element $b \in B$ such that $f(a) = b$ $p a = f^{-1}(b)$.

Now $b \in B$ \exists there exists an element $c \in C$ such that $g(b) = c$ $p b = g^{-1}(c)$.

Then $(gof)(a) = g[f(a)] = g(b) = c$ $p a = (gof)^{-1}(c) \dots (1)$

$(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(b) = a$ $p a = (f^{-1} \circ g^{-1})(1) \dots (2)$

Combining (1) and (2) we have

$$(gof)^{-1} = f^{-1} \circ g^{-1}$$

Proof of f

Theorem: If $f: A \rightarrow B$ is an invertible mapping then $f \circ f^{-1} = IB$ and $f^{-1} \circ f = IA$

Proof: f is invertible, then f^{-1} is defined by $f(a) = b \circ f^{-1}(b) = a$

where $a \in A$ and $b \in B$

Now we have to prove that $f \circ f^{-1} = IB$

Let $b \in B$ and $f^{-1}(b) = a, a \in A$

then $f \circ f^{-1}(b) = f(f^{-1}(b))$

$$= f(a) = b$$

therefore $f \circ f^{-1}(b) = b$ " $b \in B \Rightarrow f \circ f^{-1} = IB$

NOW $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$

therefore $f^{-1} \circ f(a) = a$ " $a \in A \Rightarrow f^{-1} \circ f = IA$

Hence the theorem.